Relative prices for perpetual and renewable interment rights

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30 Jul 2020

Three factors relevant to price differ between perpetual and renewable interment rights:

1. Renewable tenure requires less land for an equivalent number of interments (land-saving benefit)
2. Renewable tenure leads to end-of-tenure costs, which do not exist for perpetual rights
3. Renewable tenure allows a cemetery to continue earning new income after all the plots have been taken (perpetual income effect).

The purpose of this note is to derive a simple mathematical formula for the relationship between renewable and perpetual right prices that could be applied to any cemetery. This formula quantifies the effect of each of the three factors listed above.

1.1 How much land?

Let **T** be the length of tenure in years for a renewable right. T includes any time required to disinter the previous occupant. It is the time between burials, and not just the time that any one deceased person spends in the plot.

For the first T years of a cemetery's operation, a perpetual and renewable cemetery will require the same amount of land, because no plots will be renewed in that time.

We explore what happens from year T+1 onward, when both the perpetual and renewable right cemeteries seek to accommodate a stream of burials over time, **b(t)**.

Over the life, **L**, of a perpetual cemetery, the amount of land required is simply:

$$p\_plots(L) = \sum_{t=1}^{L} b(t)$$

However, a renewable cemetery would only require this much land over L years:

$$r\_plots(L) = \sum_{t=1}^{T} b(t) + \sum_{t=T+1}^{L} \big(b(t) - b(t-T)\big)$$

The first term, summing over the first T years, reflects the fact that renewable and perpetual cemeteries require the same amount of land initially. The second term, summing from years T+1 to L, reflects the fact that b(t – T) interments can be accommodated in existing plots vacated by earlier holders of renewable rights. Thus, the amount of land that needs to be purchased each year is less than b(t).

In the likely event that burials follow the general pattern of population growth (with a lag) and the growth rate, **g**, is fairly constant, we can simplify these equations further.

$$b(t) = b(0)(1+g)^t$$

$$i_{plots(T)} \equiv \sum_{t=1}^{T} b(t) = b(0)\sum_{t=1}^{T}(1+g)^t = b(0)(1+g)\frac{(1+g)^T - 1}{g}$$

$$p_{plots(L)} \equiv \sum_{t=1}^{L} b(t) = b(0)(1+g)\frac{(1+g)^L - 1}{g}$$

$$\frac{i\_plots(T)}{p\_plots(L)} = \frac{(1+g)^T - 1}{(1+g)^L - 1}$$

$$\frac{b(t) - b(t-T)}{b(t)} = 1 - \frac{b(t-T)}{b(t)} = 1 - \frac{b(0)(1+g)^{t-T}}{b(0)(1+g)^t} = 1 - (1+g)^{-T} = constant$$

Therefore, the relative land requirements for renewable and perpetual cemeteries facing the same constant growth rate burial profile are given by this equation:

$$\frac{r_{plots(L)} - i_{plots(T)}}{p_{plots(L)} - i_{plots(T)}} = \frac{\sum_{t=T+1}^{L}\big(b(t) - b(t-T)\big)}{\sum_{t=T+1}^{L} b(t)}$$

$$= \frac{\sum_{t=T+1}^{L}(1 - (1+g)^{-T})\, b(t)}{\sum_{t=T+1}^{L} b(t)}$$

$$= (1 - (1+g)^{-T})\frac{\sum_{t=T+1}^{L} b(t)}{\sum_{t=T+1}^{L} b(t)}$$

$$= 1 - (1+g)^{-T}$$

The land-saving benefit overall is given by:

$$LSB = \frac{r\_plots(L)}{p\_plots(L)} = \frac{r_{plots(L)} - i_{plots(T)}}{p_{plots(L)} - i_{plots(T)}}\left(1 - \frac{i\_plots(T)}{p\_plots(L)}\right) + \frac{i\_plots(T)}{p\_plots(L)}$$

$$\boxed{LSB = (1 - (1+g)^{-T})\left(1 - \frac{(1+g)^T - 1}{(1+g)^L - 1}\right) + \frac{(1+g)^T - 1}{(1+g)^L - 1}}$$

1.2 Impact of end-of-tenure costs?

At the end of a period of renewable tenure, the rights-holder must pay for the preparation of the plot for its next occupant. These end-of-tenure costs incurred in year T, **d**, will have a present value at the time the renewable right is purchased given by this formula:

$$EOT = d(1 + r)^{-T}$$

Here, **r** is the discount rate, which accounts for the time value of money. EOT must simply be added to the renewable right price that is derived from the perpetual right price, adjusted for land-saving and perpetual income effects.

1.3 Ongoing income stream for renewable cemetery?

Once the last plot is filled, a perpetual cemetery earns no further income. However, a renewable cemetery is able to earn a perpetual stream of income even after it is full because renewable uses of existing plots can continue to be sold. This perpetual income effect contributes to the price advantage that renewable tenure offers relative to perpetual tenure. Prices reflect the ratio of costs to units sold. Whereas the land saving benefit reduces renewable prices by reducing the numerator of that ratio, the perpetual income effect reduces renewable prices by increasing the denominator. We quantify that effect below.

Prices for both types of cemetery are equal to the present value of all costs divided by the present value of the number of interments sold. For a perpetual cemetery of life L,

$$PVpb(L) = \sum_{t=1}^{L} b(t)\,(1 + r)^{-t}$$

For a renewable cemetery designed to fill its last plot for the first time in year L,

$$PVrb(L) = \sum_{t=1}^{L} b(t)\,(1 + r)^{-t} + \sum_{t=L+1}^{\infty} rb(t)\,(1 + r)^{-t}$$

The new variable, rb(t), is the number of renewed burials in year t. When t > L+1 but less than L+T, then rb(t) = b(t – T). In later years, rb(t) = rb(t – T). In both cases, that is the number of plots that are becoming available again each year after having been occupied for T years.

$$PVrb(L) = PVpb(L) + \sum_{t=L+1}^{\infty} rb(t)\,(1 + r)^{-t}$$

If we continue with the assumption of a constant growth rate in burials up to year L, then we can develop a simpler expression for the infinite summation term in PVrb(L). In the notation below, k = t – L – 1.

$$\sum_{t=L+1}^{\infty} rb(t)\,(1 + r)^{-t} = \sum_{k=0}^{\infty} rb(k + L + 1)\,(1 + r)^{-(k+L+1)}$$

$$= (1+r)^{-(L+1)} \sum_{k=0}^{\infty} rb(k+L+1)(1+r)^{-k}$$

Note that under this growth assumption, rb(t) will follow a sawtooth pattern, because the upward-sloping pattern of renewable interments over the last T years before year L will be repeated infinitely after year L. It will be easier to do the infinite summation if we rearrange the terms, grouping them by the values of rb(t), instead of summing them in temporal order.

To do this, notice that we can express each value of k = 0, 1, ..., ∞ as:

$$k = m + jT$$

where j = 0, 1, ..., ∞, and m = modulo(k,T). We can express the infinite summation now as:

$$= (1+r)^{-(L+1)} \sum_{j=0}^{\infty} \sum_{m=0}^{T-1} rb(m+jT+L+1)(1+r)^{-(m+jT)}$$

Swapping the summation indices does not change the sum, so the infinite sum is:

$$= (1+r)^{-(L+1)} \sum_{m=0}^{T-1} \sum_{j=0}^{\infty} rb(m+jT+L+1)(1+r)^{-(m+jT)}$$

$$= (1+r)^{-(L+1)} \sum_{m=0}^{T-1} (1+r)^{-m} \sum_{j=0}^{\infty} rb(m+L+1)(1+r)^{-jT}$$

This simplification is possible because rb(t+jT) = rb(t), reflecting the sawtooth nature of renewed burials. Simplifying further, the infinite sum is:

$$= (1+r)^{-(L+1)} \sum_{m=0}^{T-1} (1+r)^{-m}\, rb(m+L+1) \sum_{j=0}^{\infty} (1+r)^{-jT}$$

$$= (1+r)^{-(L+1)} \sum_{m=0}^{T-1} (1+r)^{-m}\, rb(m+L+1) \left[\frac{1}{1-(1+r)^{-T}}\right]$$

$$= \frac{(1+r)^{-(L+1)}}{1-(1+r)^{-T}} \sum_{m=0}^{T-1} (1+r)^{-m}\, rb(m+L+1)$$

To make the final simplifications, note that t = m+L+1 is in the range: L+1 to L+T when m = 0 .. T-1. As noted above, when t is in that range, rb(t) = b(t – T). Therefore, the infinite sum is:

$$= \frac{(1+r)^{-(L+1)}}{1-(1+r)^{-T}} \sum_{m=0}^{T-1} (1+r)^{-m}\, b(m+L+1-T)$$

$$= \frac{(1+r)^{-(L+1)}}{1-(1+r)^{-T}} \sum_{m=0}^{T-1} (1+r)^{-m}\, b(0)(1+g)^{m+L+1-T}$$

$$= \frac{(1+r)^{-(L+1)}}{1-(1+r)^{-T}} b(0)(1+g)^{L+1-T} \sum_{m=0}^{T-1} (1+r)^{-m}\,(1+g)^{m}$$

To simplify the notation, let z = (1+g) / (1+r)

$$= \frac{b(0)\, z^{L+1}}{1-(1+r)^{-T}} (1+g)^{-T} \sum_{m=0}^{T-1} z^{m}$$

$$= \frac{b(0)\, z^{L+1-T}}{(1+r)^{T}-1} \sum_{m=0}^{T-1} z^{m}$$

Therefore, the infinite sum part of PVrb(L) is:

$$\sum_{t=L+1}^{\infty} rb(t)\,(1+r)^{-t} = \frac{b(0)\, z^{L+1-T}}{(1+r)^{T}-1} \left[\frac{1-z^{T}}{1-z}\right]$$

We can use the notation developed above to simplify the expression for PVpb(L), too:

$$PVpb(L) = \sum_{t=1}^{L} b(t)\,(1+r)^{-t}$$

$$= \sum_{t=1}^{L} b(0)\, z^{t} = b(0) z \frac{1-z^{L}}{1-z}$$

Now we can write a fairly simple expression for the perpetual income effect:

$$\boxed{PIE = \frac{PVrb(L)}{PVpb(L)} = 1 + \frac{z^{L-T}(1-z^{T})}{((1+r)^{T}-1)(1-z^{L})}}$$

1.4 Derivation of renewable price from perpetual price

Finally, we are in a position to combine the land-saving, end of tenure and perpetual income effects to express the renewable price for a single plot:

$$\boxed{Renewable\ price\ (L,T) = Perpetual\ price\ (L) \frac{LSB}{PIE} + EOT}$$

The formulae above can be used to evaluate LSB, PIE and EOT if T, L, r, g and d are known.

1.5 Does the land purchasing strategy make a difference?

The formulae above work as long as these conditions are met:

- the amount of land purchased for a renewable cemetery is LSB times the amount of land purchased for a perpetual cemetery
- land purchases occur at the same time and for the same price per square metre in the renewable cemetery and perpetual cemeteries.

These conditions could be met under a variety of possible land purchasing strategies. For example, if all the land ever needed for each type of L year-life cemetery was purchased in year 0, they would be met.

They would also be met if land were purchased just in time. That is, the perpetual cemetery could purchase land in small lots of 100 square metres and the renewable cemetery could purchase land in lots of 100xLSB square metres at the same time and for the same unit price.

The formulae derived above will work even if different lots were purchased at different times and for different prices, as long as the perpetual and renewable cemetery purchases are synchronised, the areas of land purchased are in proportion to LSB, and both cemeteries pay the same unit price as each other each time.

Given these points, the land purchasing strategy should not affect the relative prices between renewable and perpetual cemeteries, although those strategies will affect the absolute prices.