



Independent Pricing and Regulatory Tribunal

Public transport fare optimisation model

Final Report - Information Paper 8

Transport — Information Paper
May 2016

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Public transport fare optimisation model

This information paper presents the method used to estimate optimal fares. It sets out the theoretical basis and derives the key formulae. In order to apply these formulae, certain data structures are implemented within an Excel spreadsheet model. These data structures are described here. The calculation process involves the simultaneous solution of 32 linear equations, which is done in the model by a matrix inversion procedure. The methods used for predicting patronage, revenue and public transport subsidy requirements at the optimum point are described. It is shown that the second-order conditions for a local maximum of the welfare function are satisfied. Finally, equations are derived that can be used to establish the welfare result for the optimal price set – and to explore the welfare consequences of departures from these prices.

This information paper includes the following changes to the corresponding technical appendix in IPART's September 2015 Methodology Paper.

1. Formulae have been updated to reflect GST effects.
2. A new subsection has been added to explain how capacity constraints affect optimal prices.
3. A subtle point about the bounds of integration in the welfare calculation has been articulated towards the end of the welfare subsection.
4. A new subsection has been added to explain the estimation of LRMC for rail.

Key formulae

Optimal prices for public transport are here taken to be the prices that maximize a social welfare function. This version of the welfare function includes consumer and producer surplus in transport markets, as well as externalities relating to transport use (ie, congestion, air and noise pollution, accidents) and taxation. As the exclusive focus is on transport markets, this is a partial equilibrium approach. Impacts of transport on land use and the economy more broadly are ignored. Such an approach is reasonable where the emphasis of the analysis is on transport pricing, as opposed to infrastructure investment, which could have large impacts on land use.

Nomenclature:

- j “mode”: a particular combination of transport mode, time of day, and distance travelled
- λ marginal excess burden of taxation
- μ GST leakage factor¹
- p^j price for “mode” j
- e^j own-price elasticity for “mode” j
- m^i marginal external cost of “mode” i
- c^i marginal cost of “mode” i
- X^i total usage of “mode” i. Units are passenger-journeys of the specified length.

The consumer surplus part of the welfare equation includes the benefit to individuals of travel, the disbenefit of having to pay for it, and the net (dis)benefit arising from the decisions taken by other individuals about how and how much they travel (externalities). The producer surplus part of the welfare equation includes the profit earned by public transport providers, which is multiplied by a factor to take account of the excess burden of taxation required to fund a transport deficit.

Equation (1) below is derived from equation (17) in De Borger, et. al. (1996),² which sets out the derivation of the welfare relationships in detail. As their derivation is somewhat involved, it will not be repeated here. It expresses the necessary (but not sufficient) condition to guarantee a local maximum of a welfare function. It is derived by setting the partial derivative of welfare with respect to the price of “mode” j to zero³.

$$\begin{aligned} \partial W / \partial p^j &= \lambda X^j - (\partial X^j / \partial p^j)(m^j + (1+\lambda)(c^j - p^j)) \\ - \sum_{i < j} (\partial X^i / \partial p^j)(m^i + (1+\lambda)(c^i - p^i)) &= 0 \end{aligned} \quad (1)$$

¹ For every dollar of GST paid by a taxpayer in NSW, approximately 31 cents is returned by the Commonwealth to the NSW Government, implying that 69 cents “leaks.” The factor μ represents that leakage as a proportion of the GST-inclusive selling price of the relevant service: $\mu = 69\%/11$.

² Bruno De Borger, Inge Mayeres, Stef Proost and Sandra Wouters, “Optimal Pricing of Urban Passenger Transport, A Simulation Exercise for Belgium,” *Journal of Transport Economics and Policy*, January 1996, pp 31-54.

³ We have adopted the simplifying assumption that the marginal social utility of income is the same for all individuals. Alternatively one could say that this analysis proceeds with a focus on one representative individual.

De Borger et. al. did not include GST in their formulae. To include it, the following modifications to equation (1) are necessary. First, the summation in the last term must be split into the sum over public transport modes ($i < j, \text{auto}$) and automobile modes ($i = \text{auto}$) since the GST treatment differs between these mode types.

Note that p_i represents the price inclusive of GST, since that is the price to which consumers respond. Public transport tickets attract 10% GST, which is paid in the first instance to the Commonwealth. Part of that GST revenue is then paid by the Commonwealth back to the state in which it was collected. As the State Government is the proprietor of the public transport network, this pattern of payments implies that the effective sales proceeds from a public transport ticket is $p_i(1-\mu)$. The second modification to equation (1) is to substitute this expression for p_i wherever public transport prices are referred to.

No further modifications to equation (1) are needed to incorporate GST, for the following reasons. Public transport providers obtain a GST input credit on their costs, implying that the terms c_i for public transport are GST-exclusive costs. However, motorists, as final consumers of motoring inputs, do not. We make the assumption that, as motoring inputs are provided in workably competitive markets, prices and costs are approximately equal for motorists, apart from Government charges, such as fuel excise, the parking-space levy and tolls. These charges do not attract GST. The revised version of equation (1) is:

$$\begin{aligned} \partial W / \partial p_i &= \lambda X_i - (\partial X_i / \partial p_i)(m^i + (1+\lambda)(c_i - p_i(1-\mu))) \\ &- \sum_{i < j, \text{auto}} (\partial X_i / \partial p_i)(m^i + (1+\lambda)(c_i - p_i(1-\mu))) \\ &- \sum_{i = \text{auto}} (\partial X_i / \partial p_i)(m^i + (1+\lambda)(c_i - p_i)) = 0 \end{aligned}$$

In order to convert this equation into a form which is soluble for optimal prices it is necessary to assume a functional form for the demand schedule. Linear demand is assumed, with own-price elasticity equal to e_{jj} at the current set of transport prices and usage (ie, p_{i0} and X_{i0}). Specifically,

$$X_i = D + (\partial X_i / \partial p_i) p_i + \sum_{i < j} (\partial X_i / \partial p_j) p_j$$

Noting that $e_{jj} \equiv (\partial X_i / \partial p_i) (p_{i0} / X_{i0})$,

$$(\partial X_i / \partial p_i) = e_{jj} X_{i0} / p_{i0} \quad (2)$$

Also, $X_{i0} = (\partial X_i / \partial p_i) (p_{i0} / e_{jj})$, implying that

$$D = (\partial X_i / \partial p_i) p_{i0} (1 / e_{jj} - 1) - \sum_{i < j} (\partial X_i / \partial p_j) p_{j0}$$

By the assumption of linearity, the partial derivatives are constant and therefore so is D. Making this substitution and grouping price terms on the left,

$$\begin{aligned}
 & (\partial X_i / \partial p_i)(\lambda + (1+\lambda)(1-\mu)) p^i + \sum_{i < j, \text{auto}} (\lambda (\partial X_i / \partial p_i) + (\partial X_i / \partial p_i) (1+\lambda)(1-\mu)) p^i \\
 & = (\partial X_i / \partial p_i)(m^i + (1+\lambda)c^i) + \sum_{i < j, \text{auto}} (\partial X_i / \partial p_i)(m^i + (1+\lambda)c^i) \\
 & + \sum_{i=\text{auto}} (\partial X_i / \partial p_i)(m^i + (1+\lambda)(c^i - p^i)) - \lambda D \quad (3)
 \end{aligned}$$

This expression can be simplified by dividing both sides by $(\partial X_i / \partial p_i)$ and adopting the new variables:

$$Z_j \equiv (\partial X_i / \partial p_i) / (\partial X_i / \partial p_i) \text{ and}$$

$$V_j \equiv (\partial X_i / \partial p_i) / (\partial X_i / \partial p_i)$$

Note that if $i = j$

$$Z_j = V_j = 1$$

An important premise of this work is that road pricing is not able to be optimized. Notationally, the summation is over i for public transport modes ($i < \text{ax}$) and over x for automobile modes. Here, "ax" represent the range of "modes" involving the automobile mode. The "x" part of this index refers to time of day and distance components.

The automobile components of the summation on the left hand side are moved to the right hand side. Since we assume no change to automobile prices, the components of the automobile price summation involving the factor V cancel out the corresponding components of the constant D . The resulting equation (4) is shown below. Note that the final constant term is not $-\lambda D$ because the automobile terms involving $V^{\text{ax}} p^{\text{ax}_0}$ are excluded from the final summation.

$$\begin{aligned}
 \sum_{i < \text{ax}} (\lambda V_j + Z_j(1+\lambda)(1-\mu)) p^i & = \sum_{i < \text{ax}} Z_j (m^i + c^i(1+\lambda)) + \sum_x Z^{\text{ax}_j} (m^{\text{ax}} + (c^{\text{ax}} - p^{\text{ax}})(1+\lambda)) \\
 + \lambda [p^i_0 (1 - 1/e^j) + \sum_{i < j, \text{ax}} V_j p^i_0] & \quad (4)
 \end{aligned}$$

Equation (4) will be used in the calculations. This gives a set of equations, one for each non-auto mode, j .

Data structures

Index j can take up to 40 values = $5 \times 4 \times 2$. Each value is one combination of one of five transport modes (automobile, train, bus, ferry, light rail), one of four journey distances (2, 5, 15, 25 km), and one of two times of day (peak, off-peak).

Index x can take up to 8 values = 2×4 , representing the possible combinations of one of two times of day and one of four journey distances.

Index ax can take the same number of values as index x .

The marginal excess burden of taxation is a scalar value (usually $0 < \lambda < 1$).

Public transport prices are the unknowns for which we solve. This is a 32-element ($4 \times 4 \times 2$) vector.

Other 32-element vectors needed for public transport “modes” are:

- ▼ Own-price elasticities.
- ▼ Marginal costs.
- ▼ Marginal external costs.

In performing the calculations, we assume that all marginal costs and marginal external costs are constant (that is, independent of usage). To the extent this assumption is not valid, the projected welfare outcomes at prices very different from the status quo may be distorted.

For automobile “modes”, 8-element vectors are needed:

- ▼ The difference between marginal cost and the price the motorist pays ($c^{ax} - p^{ax}$);
- ▼ Marginal external costs.

Marginal external costs for each “mode” are the sum of the following components, which are each calculated separately:

- ▼ Marginal external congestion cost (mainly for automobiles, but also buses).
- ▼ Marginal external emission cost (for all modes, but less so for trains).
- ▼ Marginal external accident cost (mainly for automobiles, but also for buses and light rail).

A table of values Z_{ij} and V_{ij} is needed. Index i can take 40 values (public transport and automobiles). Index j can take only 32 values (public transport “modes” only). Table 1 below indicates how the Z values are assumed to depend on the indices i and j . Each of these indices actually represents a combination of values of three other indices: one each for time of day, distance, and mode type.

Table 1 Z values

I			J			Z
TOD	Distance	Mode	TOD	Distance	Mode	
a	b	c	a	b	c	1
a	b	c	a	v <> b	c	0
a	b	c	a	b	w <> c	Z_w^c (Based on STM runs)
a	b	c	u <> a	b	c	-0.1
a	b	c	u <> a	v <> b	c	0
a	b	c	a	v <> b	w <> c	0
a	b	c	u <> a	v <> b	w <> c	0
a	b	c	u <> a	b	w <> c	0

Note: TOD = Time of day.

Source: summary of substitution assumptions.

The first row of Table 1 states that where indices i and j are identical, Z=1. By identical we mean that the time of day = a is the same, that the distance band = b is the same, and that the mode (vehicle type) = c is the same for i and j. The last four rows of the table state that if the indices i and j differ in two or more elements (eg, different TOD and distance, different distance and mode, different TOD and mode, or all three different) then Z = 0. The two remaining cases are different mode (but same TOD and distance) or different TOD (but same distance and mode). In the former case, the Z factor is calculated based on STM runs, and in the latter case, the Z factor is assumed to be -0.1.

The V values can be derived from Z values:

$$V_{ij} = Z_i (\partial X^i / \partial p^i) / (\partial X^i / \partial p^i)$$

Using (2), this can be expressed as:

$$V_{ij} = Z_i (e^{ii} X_{i0} p_{i0}) / (e^{ij} X_{j0} p_{i0})$$

Calculation process

Equation (4) can be expressed in vector notation as:

$$A p = \text{RHS} \tag{5}$$

The diagonal elements of matrix A are equal to $(\lambda + (1+\lambda)(1-\mu))$.

The off-diagonal elements of A are $(\lambda V_{ij} + (1+\lambda)(1-\mu) Z_{ij})$.

Element j of the RHS vector is equal to

$$\sum_{i < j, \text{auto}} Z_{ij}(m^i + c^i(1+\lambda)) + \sum_{i = \text{auto}} Z_{ij}(m^i + (c^i - p^i)(1+\lambda)) \\ + \lambda[p_{j0}(1 - 1/e^{ij}) + \sum_{i < j, \text{auto}} V_{ij} p_{i0}],$$

since motoring prices are not optimized.

Equation (5) can be used to solve for the vector of optimal prices:

$$p^* = A^{-1} \text{RHS} \quad (6)$$

where A^{-1} is the inverse of matrix A .

Additional steps

Once the set of optimal prices has been determined, a series of further calculations is required.

Predicted patronage at optimum

Given prices, demand schedules for all transport modes and times of day are needed to estimate patronage. Some care may be required in taking account of subsidized travellers, such as pensioners and travellers taking advantage of the school student transport scheme.

Predicted revenue and subsidy requirement

With the knowledge of prices and patronage, farebox revenue for public transport can be estimated. Knowledge of total costs for public transport will then permit a calculation of the subsidy implications of optimal prices.

Are capacity constraints satisfied?

Optimal patronage levels for public transport must be compared to current capacity limits by mode and time of day to ensure that the transport task can actually be met.

How capacity constraints affect optimal prices

Where a mode is capacity constrained, we assume that the point (p_0, q_0) lies on the demand schedule. (ie, there is no excess demand at price p_0 and $q(p_0)$ is just equal to the constraint q_0 .) If the calculated value of $p^* > p_0$ then it implies $q^* < q_0$, in which case the capacity constraint would be relieved at the optimal price.

However, the presence of the capacity constraint precludes a situation where $p^* < p_0$ because that would imply $q^* > q_0$.

To reflect this situation, we include an additional rule that where a mode is capacity constrained, the constrained optimal price = $\text{MAX}(p_0, p^*_{\text{unconstrained}})$.

Second-order conditions are met

In order to rule out the possibility that the optimal prices calculated by this method represent a local minimum rather than a local maximum of welfare, we calculate the second derivative of the welfare function and show that it is always negative.

We start with the GST version of equation (1) for the first derivative of welfare with respect to the price for mode j:

$$\begin{aligned} \partial W / \partial p^j &= \lambda X^j - (\partial X^i / \partial p^j)(m^i + (1+\lambda)(c^i - p^i(1-\mu))) \\ &- \sum_{i < j, \text{auto}} (\partial X^i / \partial p^j)(m^i + (1+\lambda)(c^i - p^i(1-\mu))) \\ &- \sum_{i = \text{auto}} (\partial X^i / \partial p^j)(m^i + (1+\lambda)(c^i - p^i)) \end{aligned} \quad (1)$$

We assume that in the neighborhood of $\partial W / \partial p^j = 0$, demand for transport mode j is linear:

$$X^j = X_0 - \beta p^j + (p^i \text{ terms}), \text{ where } \beta > 0.$$

The terms in the demand function containing p^i (with $i < j$) will drop out when X^j is differentiated with respect to p^j . The second derivative of welfare with respect to p^j is therefore:

$$\partial^2 W / \partial p^j{}^2 = -\beta\lambda - \beta(1+\lambda)(1-\mu) = -\beta(1+2\lambda-\mu-\lambda\mu) < 0, \text{ as long as } \mu < 1.$$

Welfare implications of optimal prices

The change in welfare from moving away from optimal prices can be calculated by integrating a version of equation (1). This calculates the welfare impacts of moving a single price (for one mode, distance and time of day) away from its optimal level, while keeping other prices at their optimal levels.

$$\begin{aligned}
 \int dW &= \int \{ \lambda X_i - (\partial X_i / \partial p_i)(m^i + (1+\lambda)(c^i - p^i(1-\mu))) \\
 &\quad - \sum_{i < j, \text{auto}} (\partial X_i / \partial p_i)(m^i + (1+\lambda)(c^i - p^i(1-\mu))) \\
 &\quad - \sum_{i = \text{auto}} (\partial X_i / \partial p_i)(m^i + (1+\lambda)(c^i - p^i)) \} dp_i \\
 &= \lambda \int (X_i dp_i + p_i dX_i) - \lambda \int c_i dX_i + (1 - \mu(1+\lambda)) \int p_i dX_i - \int (m^i + c^i) dX_i \\
 &\quad - \int \sum_{i < j, \text{auto}} Z_{ij}(m^i + (1+\lambda)(c^i - p^i(1-\mu))) dX_i \\
 &\quad - \int \sum_{i = \text{auto}} Z_{ij}(m^i + (1+\lambda)(c^i - p^i)) dX_i
 \end{aligned}$$

When this indefinite integral is evaluated between the bounds p_{i0} (initial price) and p_i^* (the calculated optimal price),

$$\begin{aligned}
 \Delta W_i &= \lambda [p_i^* X_i^* - p_{i0} X_{i0} - c_i \Delta X_i] \\
 &\quad + \Delta X_i [(1 - \mu(1+\lambda)) (p_i^* + p_{i0})/2 - m^i - c^i] \\
 &\quad - \Delta X_i \sum_{i < j, \text{auto}} Z_{ij}(m^i + (1+\lambda)(c^i - p_i^*(1-\mu))) \\
 &\quad - \Delta X_i \sum_{i = \text{auto}} Z_{ij}(m^i + (1+\lambda)(c^i - p_i^*)) \quad (7)
 \end{aligned}$$

The terms in equation (7) have the following natural interpretations. The first term, if negative, represents the deadweight loss arising from raising taxes to pay for the public transport subsidy. The second term represents the sum of consumer and producer surplus in the market for mode j . The third term represents the cross-modal effects with other public transport modes. The fourth term represents the cross-modal effects with automobiles. This term expresses the welfare benefit arising from any substitution of public transport mode j for automobiles which reduces the external costs of motoring (less any contributions made by cars through road pricing.)

Before using equation (7) to estimate welfare changes, one subtle point must be observed. Since we are integrating from the global optimum point (p_i^*, p_i^*) to a point at which p_i assumes a non-optimal value while all other modes remain at their optimal prices, X_{i0} is not the level of output for mode j at which all prices are at their initial values (p_{i0}, p_{i0}) . Rather, $X_{i0} = X(p_{i0}, p_i^*)$. Similarly, $X_i^* = X(p_i^*, p_i^*)$. ΔX_i is not affected by this problem:

$$\Delta X_i = X_i^* - X_{i0} = X(p_i^*, p_{i0}) - X(p_{i0}, p_i^*) \quad (\text{because } X \text{ is linear in the } p_i)$$

In light of these points, equation (8) presents a preferable basis for the welfare calculation.

$$\begin{aligned} \Delta W_i = & \lambda [p^i X(p^i, p^i_0) - p^i_0 X(p^i_0, p^i_0) + \Delta X_i \sum_{i < j} V_j (p^i - p^i_0)] \\ & - \lambda c_i \Delta X_i + \Delta X_i [(1 - \mu(1 + \lambda)) (p^i + p^i_0) / 2 - m^i - c_j] \\ & - \Delta X_i \sum_{i < j, \text{auto}} Z_j^i (m^i + (1 + \lambda)(c^i - p^i(1 - \mu))) \\ & - \Delta X_i \sum_{i = \text{auto}} Z_j^i (m^i + (1 + \lambda)(c^i - p^i)) \quad (8) \end{aligned}$$

The third term in equation (8), involving a summation of V terms, reflects the derivation of X^i and X_{i_0} from the observed output points $X(p^i_0, p^i_0)$.

Sensitivity testing of inputs

Range testing on inputs must be done to determine how sensitive results are to input uncertainty (which affects many of the key variables).

Estimation of LRMC for rail

Unlike bus or ferry modes, the rail mode is subject to infrastructure capacity constraints that are extremely costly to remove. We estimate the Long Run Marginal Cost ("LRMC") for rail by considering the incremental demand in peak periods that could be accommodated by the next major infrastructure capacity project. That project is the second harbour rail crossing ("2HC") component of the Sydney Metro system. The difference between the LRMC and the Medium Run Marginal Cost ("MRMC") is estimated using the average incremental cost ("AIC") method:

$$LRMC - MRMC = PV(2HC \text{ capital cost} - \text{external benefits}) / PV(\Delta \text{ peak patronage}).$$

The above formula refers to present values (PV) because external benefits and the incremental patronage are achieved over time.

There are many external benefits from the construction of the 2HC, but we confine ourselves to the road construction costs that are avoided because the 2HC permits some new travellers to use rail who would otherwise have driven and contributed to the need for an expansion of road system capacity. Like the rail network, the Sydney road network is near capacity in peak periods.

We would like to use LRMC in the price optimisation formula in place of the variable c^{rail} to estimate optimal prices when rail infrastructure capacity is not fixed. Before doing so, it is necessary to confront the fact that, where road capacity is no longer fixed, the marginal external road congestion cost associated with a change in rail patronage would become zero. These points could be accommodated by making the following changes to the GST-version of equation (1), taking mode $j = \text{rail}$ (changes in *underlined italics*).

$$\begin{aligned} \partial W / \partial p^i &= \lambda X^i - (\partial X^i / \partial p^i)(m^i + (1+\lambda)(\underline{LRMC} - p^i(1-\mu))) \\ &- \sum_{i < j, \text{auto}} (\partial X^i / \partial p^i)(m^i + (1+\lambda)(c^i - p^i(1-\mu))) \\ &- \sum_{i = \text{auto}} (\partial X^i / \partial p^i)((m^i - \underline{mecc}^i) + (1+\lambda)(c^i - p^i)) = 0 \quad (9) \end{aligned}$$

The new variable mecc refers to the marginal external road congestion cost.

The original version of GST- equation (1) could be used if c^i was estimated using the following equation:

$$\begin{aligned} c^i &= \text{MRMC} + \{ 2\text{HC capital cost} \\ &- \sum_{i = \text{auto}} Z_j^i [\text{PV(avoided road construction cost)} - \text{mecc}^i / (1+\lambda)] \} \\ &/ \text{PV}(\Delta \text{ peak rail patronage}) \end{aligned}$$

This expression can be substituted into equation (9). Rearranging terms, one obtains:

$$\begin{aligned} \partial W / \partial p^i = 0 &= \lambda X^i - (\partial X^i / \partial p^i)(m^i + (1+\lambda)([\underline{MRMC} + 2\text{HCcc} / \text{PV}(\Delta \text{prp})] - p^i(1-\mu))) \\ &- \sum_{i < j, \text{auto}} (\partial X^i / \partial p^i)(m^i + (1+\lambda)(c^i - p^i(1-\mu))) \\ &- \sum_{i = \text{auto}} (\partial X^i / \partial p^i)(m^i + (1+\lambda)([\underline{\text{PV}(\text{arcc})} - \text{mecc}^i / (1+\lambda)] / \text{PV}(\Delta \text{prp}) + \underline{\text{mrmc}}^i - p^i)) \end{aligned}$$

In this case, c^i is represented by the sum of the medium run marginal cost and the ratio $2\text{HC capital cost} / \text{PV}(\Delta \text{ peak rail patronage})$.

The marginal cost of the automobile mode is the sum of the medium run marginal cost of motoring and the ratio $[\text{PV(avoided road construction cost)} \text{ less an allowance for the marginal external congestion cost that no longer applies}] / \text{PV}(\Delta \text{ peak rail patronage})$.

This equation treats the avoided road construction cost as part of the LRMC of motoring, rather than as an offset to the LRMC of rail. This is mathematically equivalent to equation (9).